QUADRATIC LIAPUNOV FUNCTIONALS FOR SYSTEMS WITH DELAY

(KVADRATICHNYE FUNKTSIONALY LIAPUNOVA DLIA SISTEM S ZAPAZDYVANIEM)

PMM Vol. 29, No. 3, 1965, pp. 564–566

Iu. M. REPIN
(Sverdlovsk)

(*Received November 19, 1964*)

The determination of a quadratic functional in explicit form for a system of equations with delay with respect to a given derivative of this functional is examined. The role of the functionals under examination is analogous to the role of Liapunov's quadratic forms for systems of ordinary differential equations. They can be applied in the calculation of quadratic criteria of behavior for transient processes in systems with delay.

1. System of equations for the functional. We will examine a system of differential equations with one constant delay of the form

$$dx \,/\, dt = Ax\,(t) + Bx\,(t - \tau) \tag{1.1}$$

Here $x\,(t)$ is an n-dimensional vector function of time, and A and B are real constant matrices. In the space of continuous vector functions $x\,(\theta)$ $(-\tau \leqslant \theta \leqslant 0)$ we define a continuous functional V in the following way:

$$\begin{aligned} V\,[x\,(\theta)] = (\alpha x\,(0),\, x\,(0)) + \int_{-\tau}^{0} (\beta\,(\theta)\, x\,(\theta),\, x\,(0))\, d\theta\, + \\ + \int_{-\tau}^{0} (\gamma\,(\theta)\, x\,(\theta),\, x\,(\theta))\, d\theta + \int_{-\tau}^{0}\int_{\theta}^{0} (\delta\,(\theta, \sigma)\, x\,(\sigma),\, x\,(\theta))\, d\sigma d\theta \end{aligned} \tag{1.2}$$

Here α is a constant symmetrical matrix, $\beta\,(\theta)$, $\gamma\,(\theta)$, $\delta\,(\theta, \sigma)$ are continuously differentiable matrix functions where $\gamma\,(\theta)$ is assumed to be symmetrical, and (x, y) denotes the scalar product of the vectors x and y.

We define the derivative of functional (1.2), by virtue of system (1.1), as

$$\frac{dV\,[x_t\,(\theta)]}{dt}\Big|_{t=+0}, \qquad x_t\,(\theta) = x\,(t + \theta)$$

Substituting $x_t\,(\theta)$ into (1.2), after differentiation of the relationship obtained with respect to t, using equations (1.1) and substituting $t = 0$ ($x_0\,(\theta)$ simply denoted $x\,(\theta)$, we obtain

$$\frac{dV}{dt}=\left(\left(\alpha A+A^*\alpha+\frac{\beta(0)+\beta^*(0)}{2}+\gamma(0)\right)x(0),\ x(0)\right)+$$
$$+((2\alpha B-\beta(-\tau))\,x(-\tau),\ x(0))-$$
$$-(\gamma(-\tau)\,x(-\tau),\ x(-\tau))+\int_{-\tau}^{0}\left(\left(A^*\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+\delta(\theta,0)\right)x(\theta),\ x(0)\right)d\theta+$$
$$+\int_{-\tau}^{0}((B^*\beta(\theta)-\delta(-\tau,\theta))\,x(\theta),\ x(-\tau))\,d\theta-\int_{-\tau}^{0}\left(\frac{d\gamma(\theta)}{d\theta}\,x(\theta),\ x(\theta)\right)d\theta-$$
$$-\int_{-\tau}^{0}\int_{\theta}^{0}\left(\left(\frac{\partial\delta(\theta,\sigma)}{\partial\theta}+\frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right)x(\sigma),\ x(\theta)\right)d\sigma\,d\theta$$

If now the functional $V[x(\theta)]$ is assumed to be unknown and the derivative functional $W[x(\theta)]$ to be given in the form

$$W[x(\theta)]=(ax(0),\ x(0))+(bx(-\tau),\ x(0))+(cx(-\tau),\ x(-\tau))+$$
$$+\int_{-\tau}^{0}(d(\theta)\,x(\theta),\ x(0))\,d\theta+\int_{-\tau}^{0}(e(\theta)\,x(\theta),\ x(-\tau))\,d\theta+$$
$$+\int_{-\tau}^{0}(f(\theta)\,x(\theta),\ x(\theta))\,d\theta+\int_{-\tau}^{0}\int_{\theta}^{0}(g(\theta,\sigma)\,x(\sigma),\ x(\theta))\,d\sigma\,d\theta \tag{1.3}$$

then a system of equations is obtained for the determination of $V[x(\theta)]$

$$\alpha A+A^*\alpha+{}^1/_2\,[\beta(0)+\beta^*(0)]+\gamma(0)=a,\qquad 2\alpha B-\beta(-\tau)=b,\qquad -\gamma(-\tau)=c$$
$$A^*\beta(\theta)-\frac{d\beta}{d\theta}+\delta(\theta,0)=d(\theta),\qquad B^*\beta(\theta)-\delta(-\tau,\theta)=e(\theta) \tag{1.4}$$
$$-\frac{d\gamma}{d\theta}=f(\theta),\qquad -\left(\frac{\partial\delta}{\partial\theta}+\frac{\partial\delta}{\partial\sigma}\right)=g(\theta,\sigma)$$

Here the matrices a, c, $f(\theta)$ are assumed to be symmetrical.

We note that if $f(\theta)\equiv 0$ and $c=0$, then $\gamma(\theta)\equiv 0$, and system (1.4) assumes the following form:

$$\alpha A+A^*\alpha+{}^1/_2\,[\beta(0)+\beta^*(0)]=a,\qquad 2\alpha B-\beta(-\tau)=b$$
$$A^*\beta(\theta)-\frac{d\beta}{d\theta}+\delta(\theta,0)=d(\theta),\qquad B^*\beta(\theta)-\delta(-\tau,\theta)=e(\theta) \tag{1.5}$$
$$-\left(\frac{\partial\delta}{\partial\theta}+\frac{\partial\delta}{\partial\sigma}\right)=g(\theta,\sigma)$$

In the general case $\gamma(\theta)$ is determined by means of simple integration and system (1.4) is transformed into form (1.5) with the right-hand side of the first equation modified in the obvious manner.

Now let us examine a possible method of solving system (1.5) assuming that $d(\theta)=e(\theta)=g(\theta,\sigma)=0$. In this case the last equation of system (1.5) gives $\delta(\theta,\sigma)=\varphi(\theta-\sigma)$, where $\varphi(\cdot)$ is a function of one variable. From the fourth equation of system (1.5) we now obtain $\varphi(-\tau-\theta)=B^*\beta(\theta)$, which is equivalent to the equality $\varphi(\theta)=B^*\beta(-\tau-\theta)$. Utilizing this equality we obtain from the third equation of the system

$$\frac{d\beta(\theta)}{d\theta} = A^*\beta(\theta) + B^*\beta(-\tau-\theta) \tag{1.6}$$

If a new unknown function $\beta_1(\theta) = \beta(-\tau-\theta)$, is introduced, then it is not difficult to become convinced that equation (1.6) is equivalent to the linear system of equations

$$\frac{d\beta(\theta)}{d\theta} = A^*\beta(\theta) + B^*\beta_1(\theta), \qquad \frac{d\beta_1(\theta)}{d\theta} = -B^*\beta(\theta) - A^*\beta_1(\theta) \tag{1.7}$$

with the condition

$$\beta(-{}^1/_2\tau) = \beta_1(-{}^1/_2\tau) \tag{1.8}$$

Thus equation (1.6) determines $\beta(\theta)$ with an accuracy to n^2 arbitrary constants. The dependence of $\beta(\theta)$ on these constants is linear. For the complete solution of system (1.5) it is now necessary to determine these n^2 constants and ${}^1/_2 n(n+1)$ constant elements of the matrix α. The first two equations of system (1.5) do give $n^2 + {}^1/_2 n(n+1)$ linear algebraic equations with $n^2 + {}^1/_2 n(n+1)$ unknowns indicated above. Leaving open the question of nonequality to zero of the determinant for this system of equations, we will examine in the next section the solution of system (1.5) for the case $n = 1$ (and some given right-hand parts).

2. Example of quadratic functional for one equation with delay. Let us assume that in (1.1) we have $n = 1$, so that A and B are real numbers; $W[x(\theta)] = -x^2(0)$.

System (1.5) then transforms into the system

$$2\alpha A + \beta(0) = -1, \qquad 2\alpha B - \beta(-\tau) = 0$$

$$A\beta(\theta) - \frac{d\beta}{d\theta} + \delta(\theta, 0) = 0, \qquad B\beta(\theta) - \delta(-\tau, \theta) = 0, \qquad \frac{\partial\delta}{\partial\theta} + \frac{\partial\delta}{\partial\sigma} = 0 \tag{2.1}$$

Just as in Section 1 we obtain $\delta(\theta, \sigma) = \varphi(\theta-\sigma)$, where $\varphi(\theta) = B\beta(-\tau-\theta)$. Substituting $\delta(\theta, 0) = B\beta(-\tau-\theta)$ into the third equation of system (2.1) we obtain an equation for the function $\beta(\theta)$

$$\frac{d\beta}{d\theta} = A\beta(\theta) + B\beta(-\tau-\theta) \tag{2.2}$$

Differentiating it with respect to θ we obtain

$$d^2\beta / d\theta^2 + k^2\beta(\theta) = 0 \qquad (k^2 = B^2 - A^2)$$

We assume that $B^2 > A^2$, then $\beta(\theta) = C_1 \cos k\theta + C_2 \sin k\theta$. Substituting this expression into equation (2.2) we obtain two homogeneous linearly dependent equations connecting C_1 and C_2

$$\begin{aligned} C_1(k - B\sin k\tau) + C_2(A - B\cos k\tau) &= 0 \\ C_1(A + B\cos k\tau) - C_2(k + B\sin k\tau) &= 0 \end{aligned} \tag{2.3}$$

Utilizing the first two equations from (2.1) we obtain

$$2\alpha A + C_1 = -1, \qquad 2\alpha B - C_1\cos k\tau + C_2 \sin k\tau = 0 \tag{2.4}$$

From (2.3) and (2.4) we determine

$$\alpha = \frac{k\sin k\tau + A\cos k\tau - B}{2k(k\cos k\tau - A\sin k\tau)} = \frac{B\sin k\tau - k}{2k(A + B\cos k\tau)}$$

$$C_1 = \frac{B(A - B\cos k\tau)}{k(k\cos k\tau - A\sin k\tau)}, \qquad C_2 = \frac{B(B\sin k\tau - k)}{k(k\cos k\tau - A\sin k\tau)}$$

Thus α and $\beta(\theta)$ are already determined (if $k \cos k\tau - A \sin k\tau \neq 0$); after elementary transformations we also obtain

$$\varphi(\theta) = B^2\left(2\alpha \cos k\theta + \frac{\sin k\theta}{k}\right)$$

We note that the equations obtained remain valid even for $B^2 < A^2$, when k is a purely imaginary number. The functional which was found may be utilized for calculating the integral quadratic criterion of behavior.

In fact, $dV[x(t+\theta)]/dt = -x^2(t)$ and (if conditions for asymptotic stability are satisfied for the examined function with delay) we have for the integral quadratic criterion

$$J = \int_0^\infty x^2(t)\,dt = \int_0^\infty -\frac{dV[x(t+\theta)]}{dt}\,dt = V[x(\theta)] \qquad (-\tau \leqslant \theta \leqslant 0) \tag{2.5}$$

Here $x(\theta)$ is the initial function of the trajectory under examination.

Selecting as the initial function

$$x_\varepsilon(\theta) = \begin{cases} 0 & (-\tau \leqslant \theta \leqslant -\varepsilon) \\ 1 + \theta/\varepsilon & (-\varepsilon \leqslant \theta \leqslant 0) \end{cases}$$

and passing to the limit $\epsilon \to 0$ in the equality, we obtain a value for the criterion of behavior for the solution $x(t)$, which is determined by initial conditions $x(\theta) = 0$ for $\theta < 0$, $x(0) = 1$

$$J = \int_0^\infty x^2(t)\,dt = \alpha = \frac{B \sin k\tau - k}{2k(A + B\cos k\tau)} \tag{2.6}$$

Finally we note that utilizing the methods of the operational calculus and the equality of Parseval for Fourier integrals in the calculation of J, we obtain the equation

$$J = \int_0^\infty x^2(t)\,dt = \frac{1}{\pi}\int_0^\infty \frac{d\omega}{(A + B\cos\omega\tau)^2 + (\omega + B\sin\omega\tau)^2} \tag{2.7}$$

Computation of the integral in (2.7) turns out to be difficult; equation (2.6) gives the value of this integral in terms of elementary functions of parameters.

Translated by B.D.